

INDIRECT ABELIAN THEOREMS AND A LINEAR VOLTERRA EQUATION

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1. Introduction and summary. We study asymptotic behavior of solutions of

$$(1.1) \quad x'(t) = k - \int_0^t [a(t-\tau) + c]x(\tau) d\tau, \quad x(0) = x_0, \quad \left(' = \frac{d}{dt} \right)$$

where k and x_0 are real, $c \geq 0$, and $a(t)$ satisfies

(H1) $a(t) \in C(0, \infty) \cap L_1(0, 1)$. $a(t)$ is nonnegative and nonincreasing, $\lim_{t \rightarrow \infty} a(t) = 0$, and $0 < a(0+) \leq \infty$;

(H2) $a(t)$ is convex downward; i.e., for $0 < \varepsilon < 1$ and $0 < t_1 < t_3 < \infty$, $\varepsilon a(t_1) + (1-\varepsilon)a(t_3) \geq a(t_2)$, where $t_2 = \varepsilon t_1 + (1-\varepsilon)t_3$.

By a familiar theorem on Volterra equations, (1.1) has a unique solution in $C^1[0, \infty)$. We define $u(t)$ as the solution of (1.1) with $k=0$, $x_0=1$, and we let $w(t) = \int_0^t u(\tau) d\tau$. It is easily checked that the solution of (1.1) is given by $x_0 u(t) + kw(t)$.

Treating (1.1) as a special case of a nonlinear equation, Levin proved in [5] that if $a(t) \in C[0, \infty)$, $a(t) \neq a(0)$, and $(-1)^k a^{(k)}(t) \geq 0$ for $0 < t < \infty$, $k=0, 1, 2, 3$, then

$$(1.2) \quad \lim_{t \rightarrow \infty} u(t) = 0$$

and

(1.3).(i) If $c + a(t) \in L_1(0, \infty)$ (in particular, $c=0$), then

$$\lim_{t \rightarrow \infty} w(t) = \left(\int_0^\infty a(t) dt \right)^{-1}$$

(ii) If $c > 0$, then $\lim_{t \rightarrow \infty} w(t) = 0$.

Levin also conjectured that

(1.4) If $c=0$, $a(t) \notin L_1(0, \infty)$, then $\lim_{t \rightarrow \infty} w(t) = 0$.

The present theorem shows that (H1) and (H2) together are nearly sufficient for (1.2), (1.3), and (1.4); in particular, Levin's conjecture is proved. The theorem also exhibits a class of kernels satisfying (H1) and (H2) but for which a different asymptotic behavior from (1.2), (1.3), and (1.4) can be established.

More specifically, note that if $a(t)$ is given by

$$(1.5) \quad \begin{aligned} (a) \quad & a(t) = \sum_{k=1}^{\infty} \delta_k \left(1 - \frac{\min\{t, kt_0\}}{kt_0} \right), \quad t_0 = \frac{2\pi}{\tau_0} > 0 \\ (b) \quad & \delta_k \geq 0, \quad 0 < \delta \equiv \sum_{k=1}^{\infty} \delta_k = a(0) < \infty \\ (c) \quad & \Omega \equiv \{k \mid \delta_k > 0\} \text{ has no common divisor } > 1, \end{aligned}$$

then $a(t) = \sum_{j=k}^{\infty} \delta_j - t \sum_{j=k}^{\infty} (\delta_j/jt_0)$, $(k-1)t_0 \leq t \leq kt_0$. It follows that $a(t)$ is continuous, and on each interval $(k-1)t_0 \leq t \leq kt_0$ it is linear with slope $-\sum_{j=k}^{\infty} (\delta_j/jt_0)$. Then $a(t)$ satisfies (H1) and (H2). When (1.5) holds, we may also have

$$(1.6) \quad \omega \equiv \sqrt{(\delta+c)} = j\tau_0, \quad j = \text{positive integer.}$$

We will establish (1.2), (1.3), and (1.4) when $a(t)$ satisfies (H1), (H2), and

(H3) $a(t)$ admits no representation (1.5) such that (1.6) holds.

Complementary to (H3) is

(H4) $a(t)$ satisfies (1.5) and (1.6).

When (H4) holds, we define $\gamma = (3\delta + 2c)/(\delta + c)$ and let

$$u_1(t) = u(t) - 2\gamma^{-1} \cos \omega t$$

and

$$w_1(t) = w(t) - 2(\gamma\omega)^{-1} \sin \omega t.$$

We prove

THEOREM. Let $c \geq 0$, and let $a(t)$ satisfy (H1) and (H2). Then

$$(i) \quad \begin{aligned} \lim_{t \rightarrow \infty} u(t) &= 0, \quad \text{if (H3) holds,} \\ \lim_{t \rightarrow \infty} u_1(t) &= 0, \quad \text{if (H4) holds.} \end{aligned}$$

(ii) If $c + a(t) \notin L_1(0, \infty)$,

$$\begin{aligned} \lim_{t \rightarrow \infty} w(t) &= 0, \quad \text{if (H3) holds,} \\ \lim_{t \rightarrow \infty} w_1(t) &= 0, \quad \text{if (H4) holds.} \end{aligned}$$

(iii) If $c + a(t) \in L_1(0, \infty)$,

$$\begin{aligned} \lim_{t \rightarrow \infty} w(t) &= \left(\int_0^{\infty} a(t) dt \right)^{-1}, \quad \text{if (H3) holds,} \\ \lim_{t \rightarrow \infty} w_1(t) &= \left(\int_0^{\infty} a(t) dt \right)^{-1}, \quad \text{if (H4) holds.} \end{aligned}$$

Generalizations of the results in [5] to nonlinear versions of (1.1) are given by Levin and Nohel in [7]. A result of Friedman (Theorem C of [3]) implies (1.2) and (1.4) for $c + a(t) = t^{-\alpha}$, $0 < \alpha < 1$. Halanay [4] studied a nonlinear equation

including (1.1) with $k=0$ when $c+a(s-\tau)-\varepsilon_0 e^{-\alpha|s-\tau|}$ is a positive kernel on $\{0 \leq s \leq t, 0 \leq \tau \leq t\}$ for all $t \geq 0$ and some $\varepsilon_0 > 0$, $\alpha > 0$.

The Laplace transform argument of our proof resembles the proofs of the "indirect abelian" theorems in [2, pp. 265-275]. Such theorems were used by Levin and Nohel in [6] to find an asymptotic expansion as $t \rightarrow \infty$ of solutions of an equation similar to (1.1) but where the kernel is, among other things, completely monotonic on $[0, \infty)$.

Throughout the discussion S denotes the subset of the complex plane given by

$$S = \{s \mid \operatorname{Re} s \geq 0, s \neq 0\}.$$

We define

$$A(s) = \lim_{T \rightarrow \infty} \int_0^T e^{-st} a(t) dt, \quad s \in S.$$

Then $A(s)$ is the Laplace transform of $a(t)$; similarly let $X(s)$ be the Laplace transform of $x(t)$. Taking Laplace transforms formally in (1.1), we obtain $X(s)p(s) = x_0 + (k/s)$, where $p(s) = (c/s) + A(s) + s$. In Lemma 5 we show that when (H3) holds, $p(s) \neq 0$ for $s \in S$. Then

$$X(s) = (x_0 + (k/s))/p(s), \quad s \in S.$$

The complex inversion formula for Laplace transforms, together with contour integration and some estimates on $A(s)$, yields

$$(1.7) \quad x(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right] e^{itt} X(i\tau) d\tau, \quad t > 0,$$

where for each $\varepsilon > 0$ the integrals are uniformly convergent in $t \geq T > 0$. The Riemann-Lebesgue theorem and other abelian arguments then yield our results.

When (H4) holds, we find that $p(s)$ has exactly the two zeros $s = \pm i\omega$ in S . A formula similar to (1.7) but with principal values taken also at $\tau = \pm \omega$ is used in this case.

§2 presents a sequence of lemmas concerning $a(t)$ and $A(s)$. The theorem is proved in §3.

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2. The Laplace transform of $a(t)$.

LEMMA 1. Let $\sigma > 0$, and let $a(t)$ satisfy (H1). Then

- (i) $e^{-\sigma t} a(t)$ satisfies (H1).
- (ii) If $a(t)$ satisfies (H2), so does $e^{-\sigma t} a(t)$.

Proof. (i) Obvious.

(ii) Since $a(t)$ is continuous, it suffices to prove the convexity relation in (H2) with $\varepsilon = 1/2$. Set $a_i = a(t_i)$ and $b_i = \exp(-\sigma t_i)$, $i = 1, 2, 3$. Then $a_i, b_i \geq 0$, $a_1 - a_2 \geq a_2 - a_3 \geq 0$, $b_1 - b_2 > b_2 - b_3 > 0$. Hence $a_1 b_1 - a_2 b_2 = a_1(b_1 - b_2) + b_2(a_1 - a_2) \geq a_2(b_2 - b_3) + b_3(a_2 - a_3) = a_2 b_2 - a_3 b_3$, and (ii) is proved.

We state without proof the following easy consequence of convexity:

LEMMA 2. If $a(t)$ satisfies (H2), then for any $\delta > 0$, the function $a(t) - a(t + \delta)$ is nonincreasing.

LEMMA 3. Let $a(t)$ satisfy (H1). Then

(i) $A(s)$ is defined, finite, and continuous in S . $A(s)$ is holomorphic in $\{\operatorname{Re} s > 0\}$.

(ii) For $\sigma + i\tau \in S$, $\tau \neq 0$,

$$(2.1) \quad \begin{aligned} |\operatorname{Im} A(\sigma + i\tau)| &\leq \int_0^{\pi/|\tau|} a(t) \sin |\tau| t \, dt \\ &\leq \int_0^{\pi/|\tau|} a(t) \, dt, \end{aligned}$$

and

$$(2.2) \quad |\operatorname{Re} A(\sigma + i\tau)| \leq \int_0^{\pi/2|\tau|} a(t) \, dt,$$

so that $|A(\sigma + i\tau)| \rightarrow 0$ as $|\tau| \rightarrow \infty$, uniformly in $0 \leq \sigma < \infty$.

(iii) If $a(t)$ also satisfies (H2) and $\sigma + i\tau \in S$, then

$$(2.3) \quad \begin{aligned} |A(\sigma + i\tau)| &\geq \frac{1}{\sqrt{2}} \int_0^{\pi/2|\tau|} \cos \tau t e^{-\sigma t} a(t) \, dt \\ &\geq \frac{1}{2\sqrt{2}} \int_0^{\pi/3|\tau|} e^{-\sigma t} a(t) \, dt \end{aligned}$$

(the case $\tau = 0$, $\tau^{-1} = \infty$, is included); and if $\tau > 0$,

$$(2.4) \quad \lim_{T \rightarrow \infty} \int_0^T a(t) \sin \tau t \, dt + \lim_{T \rightarrow \infty} \int_{\pi/2\tau}^T a(t) \cos \tau t \, dt \geq 0.$$

Proof. (i) For $s = \sigma + i\tau \in S$, $\tau > 0$, $T > 0$, define

$$\phi(T, s) = \int_0^T a(t) e^{-\sigma t} \cos \tau t \, dt$$

$$\psi(T, s) = \int_0^T a(t) e^{-\sigma t} \sin \tau t \, dt.$$

Since $a(t) \rightarrow 0$ as $t \rightarrow \infty$, $A(s)$ has a nonpositive abscissa of convergence (see e.g. [8, Chapter II]) so that $A(s)$ is holomorphic in $\{\operatorname{Re} s > 0\}$ with

$$(2.5) \quad [A(s)]^* = A(s^*) = \lim_{T \rightarrow \infty} [\phi(T, s) + i\psi(T, s)]$$

for $\operatorname{Im} s > 0$, $*$ = complex conjugate.

For any $T > 0$, ϕ and ψ are continuous functions of s . (H1) and Lemma 1(i) show that $e^{-\sigma t}a(t)$ is nonnegative and nonincreasing on $(0, \infty)$. For $s = \sigma + i\tau$, $\tau > 0$, $T_1, T_2 \geq (n + \frac{1}{2})\pi/\tau$, $n = \text{nonnegative integer}$,

$$|\phi(T_1, s) - \phi(T_2, s)| \leq \int_{(n+1/2)\pi/\tau}^{(n+3/2)\pi/\tau} a(t) dt$$

and similarly for ψ . Since $a(t) \rightarrow 0$ as $t \rightarrow \infty$, $\phi(T, s)$ and $\psi(T, s)$ converge as $T \rightarrow \infty$, uniformly in any set of the form

$$S \cap \{\sigma + i\tau \mid 0 < \tau_0 \leq \tau \leq \tau_1 < \infty\}$$

to continuous functions $\phi(s)$ and $\psi(s)$. Comparing this with (2.5), we see that (i) is proved.

(ii) Since $\operatorname{Re} A(\sigma + i\tau)$ and $\operatorname{Im} A(\sigma + i\tau)$ are respectively even and odd in τ , we may assume $\tau > 0$. Since for each $T > 0$ we have

$$\begin{aligned} |\phi(T, \sigma + i\tau)| &\leq \int_0^{\pi/2\tau} a(t) e^{-\sigma t} \cos \tau t dt \\ &\leq \int_0^{\pi/2\tau} a(t) dt, \end{aligned}$$

(2.2) holds. (2.1) is obtained similarly.

(iii) The case $\tau = 0$ is trivial, and by symmetry of $\operatorname{Re} A(\sigma + i\tau)$ and $\operatorname{Im} A(\sigma + i\tau)$ in τ , we may assume $\tau > 0$. Note first that since $A(i\tau) = \phi(i\tau) - i\psi(i\tau)$,

$$(2.6) \quad \sqrt{2}|A(i\tau)| \geq |\phi(i\tau)| + |\psi(i\tau)| \geq \phi(i\tau) + \psi(i\tau).$$

But

$$\phi(i\tau) = \int_0^{\pi/2\tau} a(t) \cos \tau t dt + \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{2(k-1)\pi/\tau}^{2k\pi/\tau} a\left(t + \frac{\pi}{2\tau}\right) \cos \left[\tau\left(t + \frac{\pi}{2\tau}\right)\right] dt$$

and

$$\psi(i\tau) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{2(k-1)\pi/\tau}^{2k\pi/\tau} a(t) \sin \tau t dt,$$

so (2.6) becomes

$$\begin{aligned} (2.7) \quad &|\sqrt{2}A(i\tau)| - \int_0^{\pi/2\tau} a(t) \cos \tau t dt \\ &\geq \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{2(k-1)\pi/\tau}^{2k\pi/\tau} \left\{ a(t) \sin \tau t + a\left(t + \frac{\pi}{2\tau}\right) \cos \left[\tau\left(t + \frac{\pi}{2\tau}\right)\right] \right\} dt. \end{aligned}$$

We note that the right-hand side of (2.7) equals the left-hand side of (2.4) and that both (2.3) for $\sigma = 0$ and (2.4) will follow if we show that this right-hand side is nonnegative. But for any integer $k \geq 1$,

$$\begin{aligned} &\int_{2(k-1)\pi/\tau}^{2k\pi/\tau} \left\{ a(t) \sin \tau t + a\left(t + \frac{\pi}{2\tau}\right) \cos \left[\tau\left(t + \frac{\pi}{2\tau}\right)\right] \right\} dt \\ &= \int_0^{\pi/\tau} \sin \tau t [a(t + x_0) - a(t + x_1) - a(t + x_2) + a(t + x_3)] dt, \end{aligned}$$

where $x_j = 2(k-1)\pi/\tau + j\pi/2\tau$, $j=0, 1, 2, 3$, and Lemma 2 with $\delta = \pi/2\tau$ shows that the integrand is nonnegative. For (2.3) with $\sigma > 0$, we apply (2.3) with $\sigma=0$ to the function $b(t) = e^{-\sigma t}a(t)$, which satisfies (H1) and (H2) by Lemma 1, and which has Laplace transform $B(s) = A(s + \sigma)$. This completes the proof of Lemma 3.

COROLLARY 3.1. *Let $a(t)$ satisfy (H1). Then $|sA(s)| \rightarrow 0$ as $s \rightarrow 0$, $s \in S$.*

Proof. We let $s = \sigma + i\tau$. Applying (2.1) and (2.2) to the function $b(t) = e^{-\sigma t}a(t)$, we have

$$|sA(s)| = |s| |B(i\tau)| \leq \sqrt{2}|s| \int_0^{\pi/|\tau|} e^{-\sigma t} a(t) dt,$$

when $\tau \neq 0$ and, trivially, also when $\tau=0$, $\sigma > 0$. Thus if $\sigma \geq |\tau|$, $|sA(s)| \leq 2\sigma \int_0^\infty e^{-\sigma t} a(t) dt = 2\sigma A(\sigma)$, while if $|\tau| > \sigma$, we have $|sA(s)| \leq 2|\tau| \int_0^{\pi/|\tau|} a(t) dt$; one of these estimates is valid for each $s \in S$. But since $a(t) \rightarrow 0$ as $t \rightarrow \infty$,

$$(2.8) \quad \int_0^x a(t) dt = o(x) \quad (x \rightarrow \infty).$$

It follows that $|\tau| \int_0^{\pi/|\tau|} a(t) dt \rightarrow 0$ as $\tau \rightarrow 0$; and (2.8) together with an elementary abelian theorem for Laplace transforms [8, p. 182] gives $\sigma A(\sigma) \rightarrow 0$ as $\sigma \rightarrow 0+$. In view of our estimates for $|sA(s)|$, the corollary is proved.

COROLLARY 3.2. *Let $a(t)$ satisfy (H1) and (H2), and suppose that $a(t) \notin L_1(0, \infty)$. Then $[s + A(s)]^{-1} \rightarrow 0$ as $s \rightarrow 0$, $s \in S$.*

Proof. By Lemma 3(iii),

$$\begin{aligned} |A(s)| &= |A(\sigma + i\tau)| \geq \frac{1}{2\sqrt{2}} \int_0^{\pi/3|\tau|} e^{-\sigma t} a(t) dt \\ &\geq m \int_0^{\pi/3|s|} a(t) dt, \end{aligned}$$

where $m^{-1} = 2\sqrt{2}e^{\pi/3}$. Then for sufficiently small $|s|$,

$$|[s + A(s)]^{-1}| \leq 2 \left(m \int_0^{\pi/3|s|} a(t) dt \right)^{-1} = o(1) \quad (|s| \rightarrow 0).$$

LEMMA 4. *Suppose $a(t)$ satisfies (H1) and (H2). Then exactly one of the following two cases holds:*

I. *Either (i) $a(0+) = \infty$ or (ii) $a(0) \equiv a(0+) < \infty$ and $\forall \tau > 0$ there exists an integer $k = k(\tau) > 0$ such that*

$$(2.9) \quad a\left(\frac{2(k-1)\pi}{\tau}\right) - 2a\left(\frac{2(k-1)\pi + \pi}{\tau}\right) + a\left(\frac{2k\pi}{\tau}\right) > 0.$$

II. (i) *There exists a positive number τ_0 and a sequence $\{\delta_k\}_{k=1}^\infty$ such that (1.5) holds.*

(ii) The numbers τ_0 , t_0 , and δ and the sequence $\{\delta_k\}$ are determined uniquely by (1.5). All positive τ such that

$$(2.10) \quad a\left(\frac{2(k-1)\pi}{\tau}\right) - 2a\left(\frac{2(k-1)\pi + \pi}{\tau}\right) + a\left(\frac{2k\pi}{\tau}\right) = 0, \quad k = 1, 2, \dots$$

are integral multiples of τ_0 .

(iii) The Laplace transform of $a(t)$ is given by

$$(2.11) \quad A(s) = \frac{\delta}{s} + \frac{1}{s^2} \sum_{k=1}^{\infty} \frac{\delta_k}{kt_0} (\exp[-skt_0] - 1), \quad s \in S.$$

Proof. First we note that a representation (1.5) implies (2.10) with $\tau = \tau_0$, so cases I and II exclude each other.

Now suppose we are not in case I, so that (2.10) holds for some $\tau = \tau_1$. Let J denote the set of positive integers j such that (2.10) holds when $\tau = \tau_1/j$. Then $1 \in J$. Also, J is a finite set; for (H1), (H2), and (2.10) with $k=1$, $\tau = \tau_1/j$, $j \in J$, show that $a(t)$ is linear with negative slope on $[0, 2j\pi/\tau_1]$ whenever $j \in J$. We let j_0 be the largest $j \in J$, and set $\tau_0 = \tau_1/j_0$. Then for any integer $j > 1$, (2.10) does *not* hold with $\tau = \tau_0/j$.

By convexity, (2.10) with $\tau = \tau_0$ shows that $a(t)$ is linear on each interval $2(k-1)\pi/\tau_0 \leq t \leq 2k\pi/\tau_0$; we let $-\lambda_k$ be its slope there. By (H1) and (H2),

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq 0, \quad \text{and} \quad \lim_{k \rightarrow \infty} \lambda_k = 0.$$

We define $t_0 = 2\pi/\tau_0$ and $\delta_k = kt_0(\lambda_k - \lambda_{k+1}) \geq 0$, $k = 1, 2, 3, \dots$. Then on the interval $(k-1)t_0 \leq t \leq kt_0$, the function defined by the right-hand side of (1.5a) has the value

$$\begin{aligned} \sum_{j=k}^{\infty} \delta_j \left(1 - \frac{t}{jt_0}\right) &= \sum_{j=k}^{\infty} (\lambda_j - \lambda_{j+1})(jt_0 - t) \\ &= \lambda_k(kt_0 - t) + \sum_{j=k+1}^{\infty} \lambda_j t_0 \\ &= \int_{\infty}^t da(\tau) = a(t). \end{aligned}$$

This proves (1.5a) and (1.5b).

For (1.5c), we note that if $j > 1$ divides all k in Ω , then

$$a(t) = \sum_{k=1}^{\infty} \delta_{jk} \left(1 - \frac{\min\{t, jkt_0\}}{jkt_0}\right),$$

and as in the proof of (1.5a), $a(t)$ is linear on $2j(k-1)\pi/\tau_0 \leq t \leq 2jk\pi/\tau_0$, $k = 1, 2, \dots$. But then (2.10) holds with $\tau = \tau_0/j$, and we chose τ_0 so as to make this impossible. This completes the proof that II(i) holds when I does not hold.

To prove (ii), suppose

$$(1.5a') \quad a(t) = \sum_{k=1}^{\infty} \delta'_k \left(1 - \frac{\min\{t, kt'_0\}}{kt'_0}\right), \quad t'_0 = 2\pi/\tau'_0$$

with corresponding (1.5b'), (1.5c'). Let $k_1 < k_2 < k_3 < \dots$ be all the elements of Ω , and $k'_1 < k'_2 < \dots$ all the elements of Ω' . By (1.5a, a'), for each i

$$(2.12) \quad k_i t_0 = k'_i t'_0 = \max \{x \mid \text{slope of } a(t) \text{ has exactly } i \text{ different values on } (0, x)\}.$$

In particular, $t_0/t'_0 = k'_1/k_1 = \text{rational number}$, so $t_0/t'_0 = p/q$, where p and q are relatively prime positive integers. Then for each i , by (2.12), $k_i p/q = k'_i = \text{integer}$. By (1.5c), $q=1$, so by (1.5c') also $p=1$ and $t_0 = t'_0$. By (1.5a, a'), $\tau_0 = \tau'_0$. By (1.5a'), the slope $-\lambda_k$ of $a(t)$ in $[(k-1)t_0, kt_0]$ is $-\sum_{j=k}^{\infty} (\delta'_j/jt_0)$, so by the definition of δ_k in the proof of (i), $\delta'_k = (\lambda_k - \lambda_{k+1})kt_0 = \delta_k$. (1.5b, b') give $\delta = \delta'$, and uniqueness is proved.

Any τ satisfying (2.10) leads, as in (i), to a representation (1.5') with $\tau'_0 = \tau/j$. By uniqueness $j\tau_0 = j\tau'_0 = \tau$, and (ii) is proved.

(iii) This follows from (1.5) by direct computation. This completes the proof of Lemma 4.

LEMMA 5. Let $a(t)$ satisfy (H1) and (H2), and let $c \geq 0$. Define

$$p(s) = (c/s) + A(s) + s, \quad s \in S.$$

Then

(i) $p(s)$ has no zeros in S if (H3) holds. If (H4) holds $p(s)$ has exactly the two zeros $s = \pm i\omega$ in S .

(ii) When (H4) holds,

$$(2.13) \quad |p(s) - \gamma(s - i\omega)| = o(|s - i\omega|) \quad (s \rightarrow i\omega, s \in S),$$

where $\gamma = (3\delta + 2c)/(\delta + c)$.

Proof. (i) First, (2.11) shows that $p(\pm i\omega) = 0$ when (H4) holds.

For $s = i\tau \neq 0$,

$$\operatorname{Re} A(i\tau) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_0^{2\pi/|\tau|} a\left(\frac{2(k-1)\pi}{|\tau|} + t\right) \cos \tau t \, dt.$$

But for each k , the integral in the sum is equal to

$$\int_0^{\pi/2|\tau|} [a(x_0 + t) - a(x_1 - t) - a(x_1 + t) + a(x_2 - t)] \cos \tau t \, dt,$$

where $x_j = [2(k-1) + j]\pi/|\tau|$, $j=0, 1, 2$. Lemma 2 with $\delta = (\pi/|\tau|) - 2t$ shows that the integrand is nonnegative. Furthermore, Lemma 4 shows that if $a(t)$ is in case I of Lemma 4 or in case II with $j\tau_0 \neq |\tau|$, $j=1, 2, 3, \dots$, then there exists k such that the integrand is positive at $t=0$, and by continuity on an interval $0 \leq t < \varepsilon$; for this k the integral is positive. We conclude that $\operatorname{Re} A(i\tau) \geq 0$, and if $\operatorname{Re} A(i\tau) = 0$, then $a(t)$ is in case II of Lemma 4 with $\tau = j\tau_0$, $j = \text{integer}$.

By Lemma 1, if $\sigma > 0$, the function $e^{-\sigma t}a(t)$ satisfies (H1) and (H2). Thus for $\sigma > 0$ (by the preceding paragraph for $\tau \neq 0$ and trivially for $\tau = 0$), $\operatorname{Re} A(\sigma + i\tau) = \int_0^\infty [e^{-\sigma t}a(t)] \cos \tau t \, dt \geq 0$.

To apply these remarks, we suppose that $p(s)=0$, $s=\sigma+i\tau \in S$. Then

$$0 = \operatorname{Re} p(s) = c\sigma/(\sigma^2 + \tau^2) + \operatorname{Re} A(\sigma + i\tau) + \sigma.$$

Therefore $\sigma=0$ and $a(t)$ is in case II with $\tau=j\tau_0$, $j=\text{integer}$. But then, using (2.11),

$$0 = \operatorname{Im} p(s) = -(c+\delta)/j\tau_0 + j\tau_0;$$

i.e., $c+\delta=(j\tau_0)^2=\tau^2$, so that (H4) holds and $s=\pm i\omega$. This proves (i).

(ii) By (2.11),

$$p(s) - s + \frac{\omega^2}{s} + \frac{\delta}{s^2} (s - i\omega) = \left[\frac{1}{s^2} \sum_{k=1}^{\infty} \frac{\delta_k}{kt_0} (\exp[-skt_0] - 1) \right] - \frac{\delta}{s^2} (s - i\omega).$$

On the left-hand side we expand s , ω^2/s , and δ/s^2 in Taylor series about $i\omega$; on the right we rearrange terms. Then

$$(2.14) \quad p(s) - \gamma(s - i\omega) + O(|s - i\omega|^2) = s^{-2} \sum_{k=1}^{\infty} \frac{\delta_k}{kt_0} (\exp[-skt_0] - 1 + skt_0 - i\omega kt_0) \quad (s \rightarrow i\omega, s \in S).$$

For $\operatorname{Re} s \geq 0$,

$$\begin{aligned} |(\exp[-skt_0] - 1 + skt_0 - i\omega kt_0)/kt_0| &\leq |\exp[-skt_0] - 1|/kt_0 + |s - i\omega| \\ &\leq 2/kt_0 + |s - i\omega|. \end{aligned}$$

On the other hand, the power series expansion of $\exp(-skt_0)$ about $s=i\omega$ yields for $k|s-i\omega| \leq 1$

$$\begin{aligned} |(\exp[-skt_0] - 1 + skt_0 - i\omega kt_0)/kt_0| &= \left| \sum_{j=2}^{\infty} [(s - i\omega)(-kt_0)]^j/j! \right| / kt_0 \\ &= k|s - i\omega|^2 \left| \sum_{j=0}^{\infty} t_0^{j+2}/(j+2)! \right| / t_0 \\ &\leq (\exp[t_0]k/t_0)|s - i\omega|^2. \end{aligned}$$

Using these two estimates, we have for $\operatorname{Re} s \geq 0$

$$(2.15) \quad \left| \sum_{k=1}^{\infty} \frac{\delta_k}{kt_0} (\exp[-skt_0] - 1 + skt_0 - i\omega kt_0) \right| \leq \frac{\exp[t_0]}{t_0} |s - i\omega|^2 \sum_{k=1}^{n(s)} k\delta_k + |s - i\omega| \sum_{k=n(s)+1}^{\infty} \delta_k + \frac{2}{t_0} \sum_{k=n(s)+1}^{\infty} \delta_k/k,$$

where $n(s)$ is the greatest integer such that $n(s)|s - i\omega| \leq 1$.

By (1.5a), $a(t) \geq \sum_{k=1}^m \delta_k [1 - (\min\{t, kt_0\})/kt_0]$ for $m=1, 2, 3, \dots$. Since

$$\int_0^{kt_0} \delta_k (1 - t/kt_0) dt = kt_0 \delta_k / 2,$$

we have $2 \int_0^{mt_0} a(t) dt \geq t_0 \sum_{k=1}^m k\delta_k$. Hence

$$(2.16) \quad |s - i\omega| \sum_{k=1}^{n(s)} k\delta_k \leq \frac{2}{t_0 n(s)} \int_0^{t_0 n(s)} a(t) dt \rightarrow 0 \quad \text{as } |s - i\omega| \rightarrow 0,$$

since $a(t) \rightarrow 0$. Also, since $\sum \delta_k < \infty$,

$$\sum_{k=n(s)+1}^{\infty} \delta_k + \frac{2}{t_0|s-i\omega|} \sum_{k=n(s)+1}^{\infty} \delta_k/k \leq (1+2/t_0) \sum_{k=n(s)+1}^{\infty} \delta_k \rightarrow 0 \quad \text{as } |s-i\omega| \rightarrow 0.$$

Combining this with (2.14), (2.15), and (2.16), we have (2.13), and Lemma 5 is proved.

3. Proof of theorem. Integrating (1.1), we obtain

$$x(t) = x_0 + kt - \int_0^t f(t-\tau)x(\tau) d\tau,$$

where $0 \leq f(t) \equiv \int_0^t [a(\tau) + c] d\tau \leq \int_0^1 a(\tau) d\tau + c + [a(1) + c]t$. By a standard result on Volterra equations [1, §7.6], $x(t)$ satisfies an inequality

$$|x(t)| \leq B_1 e^{bt}, \quad b, B_1 > 0.$$

Substituting in (1.1), we have

$$|x'(t)| \leq k + B_1 e^{bt} \int_0^1 [a(\tau) + c] d\tau + [a(1) + c] \int_0^t B_1 e^{b\tau} d\tau \leq B_2 e^{bt}.$$

Taking Laplace transforms in (1.1), we obtain $X(s)p(s) = x_0 + (k/s)$, $\operatorname{Re} s > b$, with $p(s) = c/s + A(s) + s$, as in Lemma 5. By Lemma 5(i),

$$(3.1) \quad X(s) = (x_0 + k/s)/p(s), \quad \operatorname{Re} s > b,$$

and (3.1) defines $X(s)$ as a function holomorphic in $\{\operatorname{Re} s > 0\}$ and continuous in S (for H3) or in $S - \{\pm i\omega\}$ (for H4). Also note that by (2.5) we have $[X(s)]^* = X(s^*)$; and by Lemma 3(ii), $X(\sigma + i\tau) \rightarrow 0$ as $|\tau| \rightarrow \infty$, uniformly in $0 \leq \sigma < \infty$.

(i) We set $x_0 = 1$, $k = 0$, $X = U$ in (3.1). Then

$$U(s) = \frac{1}{s} - \frac{(c/s) + A(s)}{c + sA(s) + s^2}, \quad s = \sigma + i\tau \in S.$$

For any $\sigma \geq 0$ and sufficiently large $R > 0$, the second term is in $L_1\{(-\infty, -R) \cup (R, \infty)\}$ as a function of τ , by Lemma 3(ii); and integration by parts shows that for any $T > 0$

$$\left[\int_{-\infty}^{-R} + \int_R^{\infty} \right] e^{t\tau} (\sigma + i\tau)^{-1} d\tau$$

converges uniformly for $t \geq T$. Then the exponential bound on $u(t)$ and $u(t) \in C'$ justify the inversion formula

$$(3.2) \quad 2\pi u(t) = e^{\sigma t} \int_{-\infty}^{\infty} e^{t\tau} U(\sigma + i\tau) d\tau, \quad \sigma > b, t > 0.$$

If $c + a(t) \in L_1(0, \infty)$, $A(s)$ has limit $A(0) = \int_0^{\infty} a(t) dt$ at $s = 0$, so $U(s)$ is continuous with $U(0) = 1/A(0)$. If $c + a(t) \notin L_1(0, \infty)$, Corollaries 3.1 (if $c > 0$) and 3.2 (if $c = 0$) show that $U(s) \rightarrow 0$ as $s \rightarrow 0$, $s \in S$, and again $U(s)$ is continuous at $s = 0$.

Thus if (H3) holds, Cauchy's theorem and the fact that $U(\sigma + i\tau) \rightarrow 0$ as $|\tau| \rightarrow \infty$ uniformly in σ yield

$$(3.3) \quad 2\pi u(t) = \int_{-\infty}^{\infty} e^{itt} U(i\tau) d\tau.$$

The Riemann-Lebesgue theorem for finite intervals and the uniform convergence of the integral in (3.3) yield $u(t) \rightarrow 0$ as $t \rightarrow \infty$.

On the other hand, if (H4) holds, we set

$$(3.4) \quad U_1(s) = U(s) - 2s/\gamma(s^2 + \omega^2).$$

Since $2s/\gamma(s^2 + \omega^2)$ is the Laplace transform of $2\gamma^{-1} \cos \omega t$, (3.2) holds with u_1 and U_1 in place of u and U . Using Cauchy's theorem as before, we have for $0 < \rho < \omega$,

$$\begin{aligned} 2\pi u_1(t) = & \left[\int_{-\infty}^{-\omega-\rho} + \int_{-\omega+\rho}^{\omega-\rho} + \int_{\omega+\rho}^{\infty} \right] e^{itt} U_1(i\tau) d\tau \\ & + \frac{1}{i} \left[\int_{C_\rho^-} + \int_{C_\rho^+} \right] e^{st} U_1(s) ds, \quad t > 0, \end{aligned}$$

where C_ρ^\pm is the semicircle $\{\pm i\omega + \rho e^{i\theta}, -\pi/2 \leq \theta \leq \pi/2\}$. Since $U_1(s^*) = [U_1(s)]^*$, this may also be written

$$(3.5) \quad \pi u_1(t) = \left[\int_0^{\omega-\rho} + \int_{\omega+\rho}^{\infty} \right] \operatorname{Re} \{e^{itt} U_1(i\tau)\} d\tau + \frac{1}{i} \int_{C_\rho^+} \operatorname{Re} \{e^{st} U_1(s)\} ds, \quad t > 0.$$

Note that

$$U_1(s) = \frac{1}{p(s)} - \frac{1}{\gamma(s - i\omega)} + O(1) \quad (s \rightarrow i\omega, s \in S).$$

Writing $p(s) = \gamma(s - i\omega) + [p(s) - \gamma(s - i\omega)]$, we find that

$$(3.6) \quad U_1(s) = \frac{p(s) - \gamma(s - i\omega)}{\gamma(s - i\omega)^2} \left[\frac{-1}{\gamma + (p(s) - \gamma(s - i\omega))/(s - i\omega)} \right] + O(1) \quad (s \rightarrow i\omega, s \in S).$$

Thus, for $s \in C_\rho^+$, (2.13) yields $e^{st} U_1(s) = o(\rho^{-1})$, ($\rho \rightarrow 0$). Since $|C_\rho^+| = \pi\rho$, we may let $\rho \rightarrow 0$ in (3.5) and obtain for $0 < \eta < \omega$

$$\pi u_1(t) = \left[\int_0^{\omega-\eta} + \int_{\omega+\eta}^{\infty} \right] \operatorname{Re} \{e^{itt} U_1(i\tau)\} d\tau + \lim_{\varepsilon \rightarrow 0} \left[\int_{\omega-\eta}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\omega+\eta} \right] \operatorname{Re} \{e^{itt} U_1(i\tau)\} d\tau.$$

Treating the first term as for (H3) (3.4 and integration by parts show that $\int_{\omega+\eta}^{\infty}$ converges uniformly), we have

$$(3.7) \quad \begin{aligned} \pi u_1(t) = \lim_{\varepsilon \rightarrow 0} \left[\int_{\omega-\eta}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\omega+\eta} \right] \{ [\operatorname{Re} U_1(i\tau)] \cos \tau t - [\operatorname{Im} U_1(i\tau)] \sin \tau t \} d\tau + o(1), \\ (t \rightarrow \infty), 0 < \eta < \omega. \end{aligned}$$

For real λ define

$$S(\lambda) = \sum_{k=1}^{\infty} \frac{\delta_k}{k t_0} (\sin k t_0 \lambda - k t_0 \lambda), \quad C(\lambda) = \sum_{k=1}^{\infty} \frac{\delta_k}{k t_0} (\cos k t_0 \lambda - 1).$$

Note that

$$(3.8) \quad C(\lambda) = C(-\lambda) \leq 0, \quad \lambda S(\lambda) = -\lambda S(-\lambda) \leq 0,$$

and $C(\lambda + j\omega) = C(\lambda)$, for any integer j . By (2.14),

$$p(i\tau) - i\gamma(\tau - \omega) = [-C(\tau) + iS(\tau - \omega)]/\tau^2 + O(|\tau - \omega|^2) \quad (\tau \rightarrow \omega).$$

(2.15) and the argument following it show that

$$(3.9) \quad |C(\lambda)| + |S(\lambda)| = o(\lambda), \quad \lambda \rightarrow 0.$$

Using these facts in (3.6) one computes

$$(3.10) \quad \operatorname{Re} U_1(i\tau) = \frac{-C(\tau)}{(\tau - \omega)^2 \tau^2 [\gamma^2 + o(1)]} + O(1) \quad (\tau \rightarrow \omega),$$

and

$$(3.11) \quad \operatorname{Im} U_1(i\tau) = R(\tau - \omega) + o\left(\frac{C(\tau)}{(\tau - \omega)^2}\right) + O(1) \quad (\tau \rightarrow \omega),$$

where

$$R(\lambda) = \frac{S(\lambda)[\gamma + S(\lambda)/\omega^2\lambda]}{\gamma\lambda^2\omega^2[(\gamma + S(\lambda)/\lambda\omega^2)^2 + C^2(\lambda)/\lambda^2\omega^4]}.$$

Now let $t = t^* = 2\pi/\omega$ in (3.7). Since $\sin \tau t^* = \sin(\tau - \omega)t^* = O(\tau - \omega)$, $(\tau \rightarrow \omega)$, we see from (3.9) and (3.11) that with $t = t^*$ the second term in the integrand in (3.7) is bounded on $(\omega - \eta, \omega + \eta)$. It follows from (3.7) and (3.10) that

$$(3.12) \quad \lim_{\varepsilon \rightarrow 0} \left[\int_{\omega - \eta}^{\omega - \varepsilon} + \int_{\omega + \varepsilon}^{\omega + \eta} \right] \frac{C(\tau) \cos \tau t^* d\tau}{(\tau - \omega)^2 \tau^2 [\gamma^2 + o(1)]}$$

exists and is finite. But by the choice of t^* , the integrand in (3.12) is $\leq C(\tau)/2(\tau - \omega)^2 \gamma^2 \omega^2$ for $|\tau - \omega|$ sufficiently small. Since $C(\tau) \leq 0$ we conclude

$$(3.13) \quad C(\tau)/(\tau - \omega)^2 \in L_1(\omega - \eta, \omega + \eta).$$

In view of (3.10), (3.11), and (3.13), an application of the Riemann-Lebesgue theorem to (3.7) yields

$$(3.14) \quad -\pi u_1(t) = \lim_{\varepsilon \rightarrow 0} \left[\int_{\omega - \eta}^{\omega - \varepsilon} + \int_{\omega + \varepsilon}^{\omega + \eta} \right] R(\tau - \omega) \sin \tau t d\tau + o(1), \quad t \rightarrow \infty.$$

Note that $R(-\lambda) = -R(\lambda)$. The change of variables $\lambda = \tau - \omega$ in (3.14) shows that to complete the proof of (i) we need only show that

$$(3.15) \quad r(t) \equiv \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\eta} [\sin(\omega + \lambda)t - \sin(\omega - \lambda)t] R(\lambda) d\lambda = o(1) \quad (t \rightarrow \infty).$$

A trigonometric identity and $|(\sin \lambda t)/\lambda| \leq t$ permit us to write

$$(3.16) \quad r(t) = 2 \cos(\omega t) \int_0^{\eta} R(\lambda) \sin \lambda t d\lambda.$$

To prove (3.15), note first that

$$|S'(\lambda)| = \left| \sum_{k=1}^{\infty} \delta_k (\cos kt_0 \lambda - 1) \right| \leq 2\delta.$$

Similarly, $|C'(\lambda)| \leq \delta$. Straightforward computations and estimates then show that $|R'(\lambda)| \leq M/\lambda^2$ for some constant $M < \infty$, $0 < \lambda < \eta$. Also note that $|\lambda R(\lambda)| \leq K_1 |S(\lambda)/\lambda| \leq K_2$, $K_1, K_2 < \infty$.

Now let $\varepsilon > 0$ be given; pick $\mu > 0$ so that $(M + K_2) < \mu\varepsilon/3$, and pick $T > 0$ so that $(M + K_2)/\eta T \leq \varepsilon/3$ and $|K_1 S(\lambda)/\lambda| \leq \varepsilon/3\mu$ for $0 < \lambda < \mu/T$. Then for $t \geq T$, integration by parts yields

$$\left| \int_{\mu/t}^{\eta} R(\lambda) \sin \lambda t \, d\lambda \right| \leq (M + K_2) \left(\frac{1}{\eta T} + \frac{1}{\mu} \right) \leq 2\varepsilon/3,$$

while

$$\left| \int_0^{\mu/t} R(\lambda) \sin \lambda t \, d\lambda \right| \leq \left| \int_0^{\mu/t} [\lambda R(\lambda) \sin \lambda t] / \lambda \, d\lambda \right| \leq \varepsilon/3.$$

These estimates, together with (3.16), prove (3.15) and complete the proof of (i).

(ii) We set $x_0 = 0$, $k = 1$, $X = W$ in (3.1). Note that $W(\sigma + i\tau) = O(\tau^{-2})$, $|\tau| \rightarrow \infty$, so $W(\sigma + i\tau) \in L_1\{(-\infty, R) \cup (R, \infty)\}$ for R sufficiently large and $0 \leq \sigma < \infty$.

If (H3) holds, we use Cauchy's theorem as in (i) to obtain, for each $\varepsilon > 0$ sufficiently small,

$$(3.17) \quad 2\pi w(t) = \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right] e^{itt} W(i\tau) \, d\tau + \frac{1}{i} \int_{D_\varepsilon} e^{st} W(s) \, ds,$$

where D_ε is the semicircle $\{e^{i\theta}, -\pi/2 \leq \theta \leq \pi/2\}$. Corollaries 3.1 and 3.2 show that $W(e^{i\theta}) = o(1/\varepsilon)$, ($\varepsilon \rightarrow 0$), uniformly in $-\pi/2 \leq \theta \leq \pi/2$. Since $|D_\varepsilon| = \varepsilon\pi$, we may let $\varepsilon \rightarrow 0$ in (3.17) and obtain

$$2\pi w(t) = \lim_{\varepsilon \rightarrow 0} \left[\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right] e^{itt} W(i\tau) \, d\tau, \quad t > 0.$$

Now the symmetry of $W(i\tau)$ in τ and the Riemann-Lebesgue theorem imply that for $\Delta > 0$,

$$(3.18) \quad \pi w(t) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\Delta} \operatorname{Re} \{e^{itt} W(i\tau)\} \, d\tau + o(1) \quad (t \rightarrow \infty).$$

Next we derive a formula similar to (3.18) for $w_1(t)$ in the (H4) case. Define

$$W_1(s) = W(s) - 2/\gamma(s^2 + \omega^2).$$

Note that $2/\gamma(s^2 + \omega^2)$ is the Laplace transform of $2(\gamma\omega)^{-1} \sin \omega t$. Note also that $W_1(s) = U_1(s)/s = (U_1(s)/\omega) + O(1)$, ($s \rightarrow \omega$, $s \in S$). Continuing as with $U_1(s)$ in (i) and using $W_1(i\tau) \in L_1(\omega + \eta, \infty)$, we obtain

$$\begin{aligned} \pi w_1(t) &= \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\omega - \eta} \operatorname{Re} \{e^{itt} W_1(i\tau)\} \, d\tau \\ &\quad + \omega^{-1} \lim_{\rho \rightarrow 0} \left[\int_{\omega - \eta}^{\omega - \rho} + \int_{\omega + \rho}^{\omega + \eta} \right] \operatorname{Re} \{e^{itt} U_1(i\tau)\} \, d\tau + o(1), \\ &\quad t \rightarrow \infty, 0 < \eta < \omega. \end{aligned}$$

Since $W_1(s) - W(s)$ is bounded on $(0, \omega - \eta)$, we may replace W_1 by W in the first term above. By (3.7) and conclusion (i) of the theorem, the second term is $o(1)$, $(t \rightarrow \infty)$. Thus for $0 < \Delta < \omega$,

$$(3.19) \quad \pi w_1(t) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\Delta} \operatorname{Re} \{e^{itt} W_1(i\tau)\} d\tau + o(1) \quad (t \rightarrow \infty).$$

Now set $\Delta = 1$ when (H3) holds, $\Delta = \omega/2$ when (H4) holds. By (3.18) and (3.19) we can complete the proof by showing that

$$(3.20) \quad \lim_{t \rightarrow \infty} \left(\lim_{\varepsilon \rightarrow \infty} \int_{\varepsilon}^{\Delta} \operatorname{Re} \{e^{itt} W(i\tau)\} d\tau \right) = 0.$$

If $c > 0$, $W(i\tau)$ is continuous on $[0, \Delta]$, by Corollary 3.1, so that (3.20) holds. It remains to prove (3.20) with $c = 0$.

For $0 < \tau \leq \Delta$, define

$$(3.21) \quad \begin{aligned} (a) \quad \phi_1(i\tau) &= \int_0^{\pi/2\tau} a(t) \cos \tau t \, dt \\ (b) \quad \phi_2(i\tau) &= \lim_{T \rightarrow \infty} \int_{\pi/2\tau}^T a(t) \cos \tau t \, dt \\ (c) \quad \phi(i\tau) &= \phi_1(i\tau) + \phi_2(i\tau) \\ (d) \quad \psi(i\tau) &= \lim_{T \rightarrow \infty} \int_0^T a(t) \sin \tau t \, dt. \end{aligned}$$

As in the proof of Lemma 3,

$$(3.22) \quad \operatorname{Re} A(i\tau) = \phi(i\tau) \quad \text{and} \quad \operatorname{Im} A(i\tau) = -\psi(i\tau).$$

Lemma 3 gives some useful facts about these functions. In addition, since $0 \leq a(t) \downarrow$, $\phi_2(i\tau) \leq 0$, so that (2.4) implies

$$(3.23) \quad \psi(i\tau) \geq |\phi_2(i\tau)|, \quad 0 < \tau \leq \Delta.$$

Also, (2.1), (3.22), and $0 \leq a(t) \downarrow$ yield

$$\begin{aligned} 0 \leq \psi(i\tau) &\leq \int_0^{\pi/\tau} a(t) \sin \tau t \, dt \\ &\leq 4 \int_0^{\pi/4\tau} a(t) \cos \tau t \, dt \\ &\leq 4 \int_0^{\pi/2\tau} a(t) \cos \tau t \, dt, \end{aligned}$$

i.e.,

$$(3.24) \quad 0 \leq \psi(i\tau) \leq 4\phi_1(i\tau), \quad 0 < \tau \leq \Delta.$$

The choice of Δ insures that $W(i\tau)$ is continuous on $(0, \Delta]$. Using (3.22) and our assumption $c = 0$, we compute

$$W(i\tau) = \frac{(\psi(i\tau) - \tau)}{\tau |A(i\tau) + i\tau|^2} - i \frac{\phi(i\tau)}{\tau |A(i\tau) + i\tau|^2}.$$

Defining $y(t) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\Delta} \operatorname{Re} \{e^{it\tau} W(i\tau)\} d\tau$, we have

$$y(t) = \lim_{\varepsilon \rightarrow 0} \left[\int_{\varepsilon}^{\Delta} \cos \tau t \frac{\psi(i\tau) d\tau}{\tau |A(i\tau) + i\tau|^2} - \int_{\varepsilon}^{\Delta} \frac{\cos \tau t d\tau}{|A(i\tau) + i\tau|^2} + \int_{\varepsilon}^{\Delta} \frac{\sin \tau t}{\tau} \frac{\phi(i\tau) d\tau}{|A(i\tau) + i\tau|^2} \right].$$

By Corollary 3.2, the middle integrand is continuous on $[0, \Delta]$. Since $|\sin \tau t / \tau| \leq t$ for $t > 0$ and

$$\frac{|\phi(i\tau)|}{|A(i\tau) + i\tau|} = \frac{|\phi(i\tau)|}{|\phi(i\tau) + i(\tau - \psi(i\tau))|} \leq 1,$$

Corollary (3.2) also shows that the third integrand is continuous in $[0, \Delta]$ for each t . Therefore

$$(3.25) \quad y(t) = \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{\Delta} \cos \tau t \frac{\psi(i\tau) d\tau}{\tau |A(i\tau) + i\tau|^2} - \int_0^{\Delta} \frac{\cos \tau t d\tau}{|A(i\tau) + i\tau|^2} + \int_0^{\Delta} \frac{\sin \tau t}{\tau} \frac{\phi(i\tau) d\tau}{|A(i\tau) + i\tau|^2}, \quad t > 0.$$

But for $0 < \tau < \pi/3t$ we have $\cos \tau t > 1/2$ and $\psi(i\tau) \geq 0$; hence the existence of the limit in (3.25) shows that

$$(3.26) \quad \psi(i\tau)/\tau |A(i\tau) + i\tau|^2 \in L_1(0, \Delta).$$

Applying the Riemann-Lebesgue theorem to the first two integrals in (3.25), we obtain

$$(3.27) \quad y(t) = \int_0^{\Delta} \frac{\sin \tau t}{\tau} \frac{\phi(i\tau) d\tau}{|A(i\tau) + i\tau|^2} + o(1) \quad (t \rightarrow \infty).$$

Another consequence of (3.26), together with (3.23), is

$$(3.28) \quad \phi_2(i\tau)/\tau |A(i\tau) + i\tau|^2 \in L_1(0, \Delta).$$

We rewrite (3.27) as

$$(3.29) \quad y(t) = \int_0^{\Delta} \frac{\sin \tau t}{\tau} \frac{d\tau}{\phi_1(i\tau)} + \int_0^{\Delta} \frac{\sin \tau t}{\tau} \frac{\phi_1(i\tau)\phi(i\tau) - \phi^2(i\tau) - (\psi(i\tau) - \tau)^2}{\phi_1(i\tau) |A(i\tau) + i\tau|^2} d\tau + o(1) \\ = y_1(t) + y_2(t) + o(1) \quad (t \rightarrow \infty).$$

Then

$$(3.30) \quad y_2(t) = - \int_0^{\Delta} \sin \tau t \left[\frac{\phi_1(i\tau) + \phi_2(i\tau)}{\phi_1(i\tau)} \phi_2(i\tau) + \frac{\psi(i\tau) - \tau}{\phi_1(i\tau)} (\psi(i\tau) - \tau) \right] \frac{d\tau}{\tau |A(i\tau) + i\tau|^2}.$$

Now, (3.23) and (3.24) show that $|(\phi_1(i\tau) + \phi_2(i\tau))/\phi_1(i\tau)|$ and $|\psi(i\tau)/\phi_1(i\tau)|$ are bounded on $(0, \Delta)$. Since $a(t) \notin L_1(0, \infty)$, (3.21a) gives

$$(3.31) \quad \phi_1(i\tau) \geq \frac{1}{2} \int_0^{\pi/3\tau} a(t) dt \rightarrow \infty, \quad \text{as } \tau \rightarrow 0,$$

so that $|\tau/\phi_1(i\tau)|$ is also bounded. Then by (3.26), (3.28), and Corollary 3.2, the

coefficient of $\sin \tau t$ in (3.30) is in $L_1(0, \Delta)$. Using the Riemann-Lebesgue theorem once more, we have

$$(3.32) \quad \lim_{t \rightarrow \infty} y_2(t) = 0.$$

Finally, to treat $y_1(t)$, we note that for $\tau > 0$

$$\frac{d}{d\tau} \phi_1(i\tau) = - \int_0^{n/2\tau} t a(t) \sin \tau t \, dt \leq 0.$$

Thus by (3.31), $1/\phi_1(i\tau) \downarrow 0$ as $\tau \downarrow 0$; in particular, $1/\phi_1(i\tau)$ is of bounded variation on $[0, \Delta]$, and a familiar theorem concerning the kernel $(\sin \tau t)/\tau$ [8, p. 64] yields $y_1(t) \rightarrow 0$ as $t \rightarrow \infty$. Combining this with (3.29) and (3.32), we have (3.20), and (ii) is proved.

(iii) A proof similar to that for (ii) can be obtained by considering $W_2(s) \equiv W(s) - 1/A(0)s$. We obtain (iii) from (i) as done in the proof of Theorem 2(i) of [5].

By the definition of $w(t)$, we have $u(t) = w'(t)$, where the asymptotic behavior of $u(t)$ is given by conclusion (i). (1.1) for w becomes

$$(3.33) \quad u(t) - 1 = - \int_0^t a(t-\tau) w(\tau) \, d\tau,$$

since $c + a(t) \in L_1(0, \infty)$ implies $c = 0$.

When (H3) holds, Levin's proof of Theorem 2(i) in [5] applies word for word to give $w(t) \rightarrow (\int_0^\infty a(t) \, dt)^{-1}$ as $t \rightarrow \infty$.

If (H4) holds, we use $A(i\omega) = -i\omega$ (from Lemma 5) and $a(t) \in L_1(0, \infty)$ to compute

$$\begin{aligned} - \int_0^t a(t-\tau) \frac{2 \sin \omega \tau \, d\tau}{3\omega} &= \frac{-2}{3\omega} \int_0^t a(\tau) \sin [\omega(t-\tau)] \, d\tau \\ &= \operatorname{Im} \left\{ \frac{-2e^{i\omega t}}{3\omega} \left[A(i\omega) - \int_t^\infty a(\tau) e^{-i\omega \tau} \, d\tau \right] \right\} \\ &= \frac{2 \cos \omega t}{3} + o(1) \quad (t \rightarrow \infty). \end{aligned}$$

Then by (3.33) and conclusion (i),

$$\int_0^t a(t-\tau) w_1(\tau) \, d\tau = 1 + o(1) \quad (t \rightarrow \infty),$$

and $w_1'(t) \rightarrow 0$ as $t \rightarrow \infty$. The proof of (iii) can now be completed by the method of Theorem 2(i) of [5], which gives $w_1(t) \rightarrow (\int_0^\infty a(t) \, dt)^{-1}$ as $t \rightarrow \infty$. This completes the proof of the theorem.

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